AD-A138 946 A NEW EIGENFUNCTION EXPANSION AND ITS APPLICATION TO WAVEGUIDE ACOUSTICS(U) PANNATONI (RONALD F) MORRISTOWN NJ R F PANNANTONI 15 FEB 84 N00014-83-C-0230 1/1

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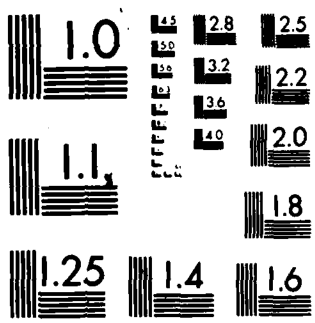

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AD A138946

Report N00014-83-C-0230-TR-1

A NEW EIGENFUNCTION EXPANSION AND ITS APPLICATION TO WAVEGUIDE ACOUSTICS

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15 FEB 1984

Interim Technical Report
for Period 15 JUN 1983 to 15 DEC 1983

Prepared for
OFFICE OF NAVAL RESEARCH
800 N. Quincy Street
Arlington, Virginia 22217

DCASMA SPRINGFIELD
240 Route 22
Springfield, New Jersey 07081

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER N00014-83-C-0230-TR-1	2. GOVT ACCESSION NO. AD-A138946	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A NEW EIGENFUNCTION EXPANSION AND ITS APPLICATION TO WAVEGUIDE ACOUSTICS		5. TYPE OF REPORT & PERIOD COVERED Interim technical report 15 JUN 1983 - 15 DEC 1983
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ronald F. Pannatoni		8. CONTRACT OR GRANT NUMBER(s) N00014-83-C-0230
9. PERFORMING ORGANIZATION NAME AND ADDRESS DR. RONALD F. PANNATONI 36-5B Farmhouse Lane Morristown, New Jersey 07960		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 386-965
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE OF NAVAL RESEARCH, CODE 425UA 800 N. Quincy Street Arlington, Virginia 22217		12. REPORT DATE 15 FEB 1984
		13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) DCASMA SPRINGFIELD 240 Route 22 Springfield, New Jersey 07081		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release. Distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

(1) Acoustic propagation
(2) Boundary value problems
(3) Coupled mode theory
(4) Eigenfunction expansions
(5) Eigenvalue problems
(6) Normal mode theory
(7) Separation of variables
(8) Waveguide propagation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes the application of new eigenfunction expansions to the analysis of sound propagation in a two-dimensional waveguide. One boundary of the waveguide is flat, the other boundary is uneven, and the waveguide is filled with inhomogeneous, lossless fluid. The expansions accommodate the conditions that the pressure vanish at the flat boundary and that the normal component of fluid velocity vanish at the uneven boundary. cont

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S/N 0102-LF-014-6601

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The method of expansion is novel in that two essentially independent functions are expanded simultaneously. Expansions of this kind provide a complete representation of the exact pressure and velocity fields at the boundaries of the waveguide as well as in the fluid. Conditions at the uneven boundary can be satisfied because the eigenvalue problem that generates the expansion functions has a boundary condition that contains the eigenvalue. The eigenvalues are therefore complex even though the waveguide is lossless.

The coefficients in the field expansions vary along the direction of propagation. A linear system of first order ordinary differential equations governs these coefficients. The equations are almost identical to coupled mode equations for waveguides that have parallel plane boundaries. The coefficients are also discontinuous at the locations of line sources in the waveguide.

1. INTRODUCTION

This report describes the application of a new type of eigenfunction expansion to the analysis of sound propagation in a two-dimensional waveguide. In the present discussion one boundary of the waveguide is flat, the other boundary is uneven, and the waveguide is filled with inhomogeneous, lossless fluid. The expansions are designed to accommodate the conditions that the pressure vanish at the flat boundary and that the normal component of fluid velocity vanish at the uneven boundary.

The method of expansion is novel in that two essentially independent functions are expanded simultaneously. Expansions of this kind provide a complete representation of the exact pressure and velocity fields at the boundaries of the waveguide as well as in the fluid. Conditions at the uneven boundary can be satisfied because the eigenvalue problem that generates the expansion functions has a boundary condition that contains the eigenvalue. This boundary condition is the key to our analysis. It causes the eigenvalues to be complex even though the waveguide is lossless.

The coefficients in the field expansions vary along the direction of propagation. These coefficients satisfy a linear system of first order ordinary differential equations that are almost identical to coupled mode equations for waveguides that have parallel plane boundaries. The coefficients are also discontinuous at positions associated with line sources in the waveguide.

To emphasize the treatment of the uneven boundary, we assume in the present discussion that the fluid density is a continuous function of depth alone. This

restriction is not fundamental to our methods, however. In our next report we shall treat lateral variations in density as well.

The present report is organized as follows. First, we discuss the general eigenvalue problem and state the principal eigenfunction expansion theorem associated with it (Section 2). Although the eigenfunctions are not orthogonal in any classical sense, there is a simple expression in closed form for the expansion coefficients. Next, we begin discussion of the waveguide problem with a description of its mathematical formulation (Section 3). Eigenvalue problems of the kind reviewed at the outset of this report originate in an asymptotic analysis of self-sustaining oscillations in the waveguide (Section 4). These problems generate the eigenfunctions that we use to represent the exact sound fields in the waveguide. Finally, we derive a linear system of first order ordinary differential equations along with jump conditions that govern the coefficients in the eigenfunction expansions of the exact sound fields (Section 5).

2. SURVEY OF THE EIGENVALUE PROBLEM

The eigenvalue problem associated with self-sustaining oscillations in the waveguide is a two-point boundary value problem for a second order ordinary differential equation on a finite interval. The eigenvalue appears quadratically in the differential equation and linearly in one boundary condition. The differential equation has the form

$$\rho(z) \frac{d}{dz}\left(\frac{1}{\rho(z)} \frac{d}{dz} \phi(z) \right) + \left(K^2(z) - k^2 \right) \phi(z) = 0 \tag{1}$$

on the open interval $-H < z < 0$. We require that the functions $\rho(z)$ and $K(z)$ take only real values, and that $\rho(z) > 0$. In addition, we assume that $d\rho(z)/dz$ and $K(z)$ are continuous on the closed interval $-H \leq z \leq 0$. The boundary conditions are

$$\frac{d}{dz}\phi(-H) + \dot{H}\, i\, k\, \phi(-H) = 0 \ , \tag{2}$$

$$\phi(0) = 0 \ . \tag{3}$$

The parameter $\dot{H}$ may take any real value.

A detailed study of the spectrum of eigenvalues k will be presented in the next report. Here we briefly note some of the main features of the spectrum. First of all, it is discrete. We may list the eigenvalues as k_n, where the index n takes any value among the positive and negative whole numbers ±1, ±2, and so on. Second, all but at most a finite number of the eigenvalues are purely imaginary. The distribution of those eigenvalues having nonvanishing real parts is symmetric with respect to the imaginary axis in the complex k plane. These eigenvalues are bounded in magnitude by the maximum value that the function $|K(z)|$ attains on the closed interval.

$$|k_n| \leq \max_{-H \leq z \leq 0} |K(z)| \quad \text{if} \quad \mathrm{Re}(k_n) \neq 0 \ . \tag{4}$$

If $\dot{H} \neq 0$ then the imaginary parts of these eigenvalues do not vanish. This is clear from the identity

$$\mathrm{Im}(k_n) = \frac{\dot{H}\,|\phi^2(-H)|}{2\int_{-H}^{0} |\phi^2(z)|\,\frac{\rho(-H)}{\rho(z)}\,dz} \quad \text{if} \quad \mathrm{Re}(k_n) \neq 0 \;, \tag{5}$$

and the fact that $\phi_n(-H) \neq 0$; for condition (2) would force the eigenfunction to vanish identically if $\phi_n(-H)$ were zero. Finally, the purely imaginary eigenvalues k_n tend to be spaced almost periodically at intervals of π/H along the imaginary axis in the complex k plane when the magnitude of k_n is large.

The eigenvalues can be listed in a way that makes description of some of these features easy. For example, the eigenvalues with nonvanishing real parts can be grouped together. There is an even number, say 2ν, of such eigenvalues, and the listing can be constructed so that $Re(k_n) \neq 0$ if and only if $|n| \leq \nu$. In addition, it is convenient to order the eigenvalues of this kind so that

$$k_{-n} = -k_n^* \quad \text{if} \quad \mathrm{Re}(k_n) \neq 0 \;, \tag{6}$$

where the asterisk denotes the complex conjugate. The purely imaginary eigenvalues can be listed with regard to the trend among them eventually to form two infinite, almost-periodic sequences:

$$k_n = i[\theta + (n - 0.5)\pi]/H + O(1/n) \quad \text{as} \quad n \to +\infty \;, \tag{7}$$

$$k_n = i[\theta + (n + 0.5)\pi]/H + O(1/n) \quad \text{as} \quad n \to -\infty \;. \tag{8}$$

The angle θ is determined by the parameter $\dot{H}$.

$$\theta = \arctan(\dot{H}) \quad \text{where} \quad |\theta| < \pi/2 \ . \tag{9}$$

A simple analysis explains the asymptotic behavior of the purely imaginary eigenvalues and of the corresponding eigenfunctions. For large $|k_n|$ we introduce approximate solutions $\hat{k}_n$ and $\hat{\phi}_n(z)$ such that

$$k_n = \hat{k}_n + O(\hat{k}_n^{-1} H^{-2}) \ , \tag{10}$$

$$\phi_n(z) = \left(\frac{\rho(z)}{\rho(-H)}\right)^{1/2} \hat{\phi}_n(z) + O\left(\max |\hat{\phi}_n(z)| \, \hat{k}_n^{-1} H^{-1}\right) \ . \tag{11}$$

The lowest order approximation to problem (1-3) in terms of $\hat{k}_n$ and $\hat{\phi}_n(z)$ has the same form as if $\rho(z)$ were constant and $K(z)$ vanished identically:

$$\frac{d^2}{dz^2} \hat{\phi}_n(z) - \hat{k}_n^2 \hat{\phi}_n(z) = 0 \ , \tag{12}$$

$$\frac{d}{dz} \hat{\phi}_n(-H) + \dot{H} i \hat{k}_n \hat{\phi}_n(-H) = 0 \ , \tag{13}$$

$$\hat{\phi}_n(0) = 0 \ . \tag{14}$$

Thus,

$$\hat{k}_n = i \hat{\kappa}_n \ , \tag{15}$$

$$\hat{\phi}_n(z) = A_n \sin(\hat{\kappa}_n z) \ , \tag{16}$$

where the amplitude $A_n \neq 0$ is arbitrary, and

$$\hat{\kappa}_n = [\theta + (n - 0.5)\pi]/H \quad \text{if} \quad n = 1, 2, \ldots , \tag{17}$$

$$\hat{\kappa}_n = [\theta + (n + 0.5)\pi]/H \quad \text{if} \quad n = -1, -2, \ldots . \tag{18}$$

Equations (10-11) and (15-18) therefore account for the approximate periodicity of the purely imaginary eigenvalues, and they also indicate that the corresponding eigenfunctions are basically sinusoidal. These features suggest that there might be enough redundancy among the eigenfunctions for them to be used to represent two functions at the same time. For example, if $\dot{H} = 0$ then the spectrum can be divided into two disjoint parts, and the eigenfunctions associated with each part can be used to construct a unique representation of a single function. In general such a division appears not to be convenient. We shall see, however, that the collection of eigenfunctions can always be easily used as a whole to construct unique representations of two functions simultaneously.

Orthogonality usually plays a role in theories of eigenfunction expansions. There is no orthogonality relation in any classical sense associated with the eigenfunctions of problem (1-3) if $\dot{H} \neq 0$, however. In fact, any one of these eigenfunctions can be expressed infinitely many ways in terms of the others. Nevertheless, in the theory of these eigenfunctions there is a relation involving both the eigenfunctions and also the eigenvalues which is somewhat similar to orthogonality. In conjunction with certain convergence properties of the eigenfunction expansions, this relation can be used to calculate the expansion coefficients. It may therefore be regarded as a natural substitute for the lacking

orthogonality principle. We state it in the next theorem.

Theorem.

Let k_m and k_n be eigenvalues of problem (1-3), and let $\phi_m(z)$ and $\phi_n(z)$ be the corresponding eigenfunctions.

(a) If $k_m \neq k_n$ then

$$(k_m + k_n)\int_{-H}^{0} \phi_m(z)\,\phi_n(z)\,\frac{\rho(-H)}{\rho(z)}\,dz - \dot{H}\,i\,\phi_m(-H)\,\phi_n \quad) = 0 \;. \tag{19}$$

(b) For all but at most a finite number of indices n,

$$2\,k_n \int_{-H}^{0} \phi_n^2(z)\,\frac{\rho(-H)}{\rho(z)}\,dz - \dot{H}\,i\,\phi_n^2(-H) \neq 0 \;. \tag{20}$$

To derive relation (19) let us first express eq.(1) in operator form $L(\phi,k) = 0$. We then obtain the following standard identity by partial integration of the difference $(\phi_m/\rho)L(\phi_n,k_n) - (\phi_n/\rho)L(\phi_m,k_m)$:

$$(k_m^2 - k_n^2)\int_{-H}^{0} \phi_m(z)\,\phi_n(z)\,\frac{1}{\rho(z)}\,dz$$

$$= \left\{\phi_n(z)\,\frac{1}{\rho(z)}\,\frac{d}{dz}\,\phi_m(z) - \phi_m(z)\,\frac{1}{\rho(z)}\,\frac{d}{dz}\,\phi_n(z)\right\}_{z=-H}^{z=0} \;. \tag{21}$$

Application of boundary conditions (2-3) simplifies the right side of this identity.

$$(k_m^2 - k_n^2) \int_{-H}^{0} \phi_m(z)\,\phi_n(z)\, \frac{1}{\rho(z)}\, dz$$

$$= (k_m - k_n)\, \dot{H}\, i\, \phi_m(-H)\,\phi_n(-H)\, \frac{1}{\rho(-H)} \quad . \tag{22}$$

Then division by the factor $(k_m - k_n) \neq 0$ yields eq.(19).

To derive relation (20) let us first observe from the asymptotic approximations (10-11) that

$$2\, k_n \int_{-H}^{0} \phi_n^2(z)\, \frac{\rho(-H)}{\rho(z)}\, dz \;-\; \dot{H}\, i\, \phi_n^2(-H)$$

$$= 2\, \hat{k}_n \int_{-H}^{0} \hat{\phi}_n^2(z)\, dz \;-\; \dot{H}\, i\, \hat{\phi}_n^2(-H) \;+\; O\left\{ |A_n^2|/n \right\} \quad . \tag{23}$$

as $n \to \pm\infty$. Evaluation of the right side of this approximation by use of solutions (15-18) is straightforward.

$$2\, \hat{k}_n \int_{-H}^{0} \hat{\phi}_n^2(z)\, dz \;-\; \dot{H}\, i\, \hat{\phi}_n^2(-H) \;=\; \hat{k}_n\, H\, A_n^2 \quad . \tag{24}$$

Equations (23-24) imply eq.(20) since $|\hat{k}_n| \to \infty$ as $n \to \pm\infty$.

We conclude this section with a statement of the principal eigenfunction expansion theorem associated with problem (1-3). For a pair of essentially independent functions $f(z)$ and $g(z)$ on the closed interval $-H \leq z \leq 0$,

this theorem asserts the existence and the uniqueness of eigenfunction expansions of the form

$$f(z) = \sum_{n=-\infty}^{\infty}{}' \; a_n \, \phi_n(z) \quad , \tag{25}$$

$$g(z) = \sum_{n=-\infty}^{\infty}{}' \; a_n \, i \, k_n \, \phi_n(z) \quad . \tag{26}$$

(The accent indicates that a sum has no $n = 0$ term, in accord with our way of listing the solutions of the eigenvalue problem.) The coefficients a_n appearing in these expansions are determined jointly by the functions $f(z)$ and $g(z)$. The expansions are valid simultaneously; the same coefficients a_n appear in both of them. We may therefore regard the eigenfunction expansions (25-26) as a representation of the complex vector valued function $(f(z),g(z))$ in terms of the nonorthogonal basis elements $(\phi_n(z), i\,k_n\,\phi_n(z))$.

In the statement of this theorem we shall assume that relation (20) is valid for all n. This assumption is not fundamental to our theory. Indeed, in the next report we shall discuss some cases that violate it, and we shall state and prove the eigenfunction expansion theorems that hold for these cases. To simplify the introductory discussion of our expansion methods, however, we have chosen to avoid these complications here.

The Principal Eigenfunction Expansion Theorem.

Suppose that the function $f(z)$ has a continuous third derivative, and that the function $g(z)$ has a continuous

second derivative. Suppose that these functions also satisfy the boundary conditions

$$\frac{d}{dz} f(-H) + \dot{H}\, g(-H) = 0 \,, \tag{27}$$

$$f(0) = 0 \,, \tag{28}$$

$$g(0) = 0 \,. \tag{29}$$

If relation (20) is valid for all n in the realization of problem (1-3) that generates the eigenfunctions, then there exist unique eigenfunction expansions of the form (25-26) for the functions $f(z)$ and $g(z)$. Furthermore, the expansion for $f(z)$ can be differentiated twice, and the expansion for $g(z)$ can be differentiated once. Parts (a) and (b) of the theorem make these statements precise.

(a) Existence of the eigenfunction expansions.

Define the coefficients a_n by means of the formula

$$a_n = \frac{\displaystyle\int_{-H}^{0} [k_n f(z) - i\, g(z)]\, \phi_n(z) \frac{\rho(-H)}{\rho(z)}\, dz - \dot{H}\, i\, f(-H)\, \phi_n(-H)}{\displaystyle 2\, k_n \int_{-H}^{0} \phi_n^2(z) \frac{\rho(-H)}{\rho(z)}\, dz - \dot{H}\, i\, \phi_n^2(-H)}$$

for $n = \pm 1, \pm 2$, and so on . (30)

(The assumption about relation (20) prevents division by zero in this formula.)

Define two sequences of functions $\{f_N(z)\}$ and $\{g_N(z)\}$ in terms of these coefficients by means of the symmetric partial sums

$$f_N(z) = \sum_{n=-N}^{N}{}' a_n \phi_n(z) , \tag{31}$$

$$g_N(z) = \sum_{n=-N}^{N}{}' a_n i k_n \phi_n(z) . \tag{32}$$

Then the following sequences of functions converge uniformly to the indicated limits on the closed interval $-H \leq z \leq 0$ as $N \to \infty$:

$$f_N(z) \to f(z) , \quad g_N(z) \to g(z) \quad \text{and} \quad \frac{d}{dz} f_N(z) \to \frac{d}{dz} f(z) . \tag{33}$$

Also, the following sequences of functions converge boundedly to the indicated limits on the open interval $-H < z < 0$ as $N \to \infty$:

$$\frac{d}{dz} g_N(z) \to \frac{d}{dz} g(z) \quad \text{and} \quad \frac{d^2}{dz^2} f_N(z) \to \frac{d^2}{dz^2} f(z) . \tag{34}$$

(b) Uniqueness of the eigenfunction expansions.

Given a sequence of complex numbers $\{\hat{a}_n\}$, where $n = \pm 1, \pm 2$, and so on, use these numbers to construct two sequences of functions $\{\hat{f}_N(z)\}$ and $\{\hat{g}_N(z)\}$ by means of the symmetric partial sums

$$\hat{f}_N(z) = \sum_{n=-N}^{N}{}' \hat{a}_n \phi_n(z) , \tag{35}$$

$$\hat{g}_N(z) = \sum_{n=-N}^{N}{}' \hat{a}_n \, i k_n \phi_n(z) . \tag{36}$$

Suppose that as $N \to \infty$,

$$\hat{f}_N(-H) \to f(-H) , \tag{37}$$

$$\int_{-H}^{0} \hat{f}_N(z) \, h(z) \, dz \to \int_{-H}^{0} f(z) \, h(z) \, dz , \tag{38}$$

$$\int_{-H}^{0} \hat{g}_N(z) \, h(z) \, dz \to \int_{-H}^{0} g(z) \, h(z) \, dz . \tag{39}$$

whenever the test function $h(z)$ has a continuous derivative. Then $\hat{a}_n = a_n$ for all n. Indeed, if $\hat{a}_n = a_n$ for all n, then conditions (37-39) are satisfied since $f_N(z) \to f(z)$ and $g_N(z) \to g(z)$ uniformly on the closed interval. These convergence conditions therefore completely characterize the eigenfunction expansions.

The statements in the existence part of the theorem will be proved in the next report. The uniqueness of the eigenfunction expansions is easy to prove, however, and the proof motivates formula (30) for the expansion coefficients. Thus, assuming that conditions (37-39) are satisfied, we base this proof on consideration of the following quantity:

$$\int_{-H}^{0} [k_m \hat{f}_N(z) - i\,\hat{g}_N(z)]\,\phi_m(z)\frac{\rho(-H)}{\rho(z)}\,dz - \dot{H}\,i\,\hat{f}_N(-H)\,\phi_m(-H)\ . \quad (40)$$

It is clear from definitions (35-36) that this equals the partial sum

$$\sum_{n=-N}^{N}{}' \hat{a}_n \left\{ (k_m + k_n) \int_{-H}^{0} \phi_m(z)\,\phi_n(z)\frac{\rho(-H)}{\rho(z)}\,dz - \dot{H}\,i\,\phi_m(-H)\,\phi_n(-H) \right\} . \quad (41)$$

In view of relation (19), only the n = m term can contribute anything nonzero to this partial sum. Thus, when m is fixed the limit of the partial sum as $N \to \infty$ is equal to the value of this term. Also, using condition (37) and taking $h(z) = \phi_m(z)/\rho(z)$ in conditions (38-39), we find that the limit of quantity (40) as $N \to \infty$ can be expressed in the same form as this quantity, but with the functions $\hat{f}_N$ and $\hat{g}_N$ replaced by f and g, respectively. Equating these limits, we find that

$$\int_{-H}^{0} [k_m\, f(z) - i\, g(z)]\,\phi_m(z)\frac{\rho(-H)}{\rho(z)}\,dz - \dot{H}\,i\,f(-H)\,\phi_m(-H)$$

$$= \hat{a}_m \left\{ 2\,k_m \int_{-H}^{0} \phi_m^2(z)\frac{\rho(-H)}{\rho(z)}\,dz - \dot{H}\,i\,\phi_m^2(-H) \right\} . \quad (42)$$

Comparison of this result with formula (30) shows that $\hat{a}_m = a_m$ as claimed.

3. FORMULATION OF THE WAVEGUIDE PROBLEM

We now begin to consider the acoustics problem outlined in the introduction. The objective in this problem is to find how a waveguide filled with fluid responds to small amplitude excitations from a source in its interior.

We let the top boundary of the waveguide be a free surface that coincides in equilibrium with a horizontal plane of infinite extent. One direction in this plane is called range and denoted by the parameter x. All features of the problem are translationally invariant along the horizontal direction perpendicular to the range. The vertical direction is denoted by the parameter z, and $z = 0$ on the top boundary.

The bottom boundary of the waveguide is rigid and uneven. The depth of the bottom varies with range according to an equation of the form $z = -H(x)$, where the depth function $H(x)$ is strictly positive and has a continuous second derivative.

The fluid in the waveguide is inhomogeneous but lossless. The local mass density of the fluid varies only with depth, and it is described by the function $\rho(z)$. The local sound speed in the fluid may vary with range as well as with depth, however, and it is described by the function $c(x,z)$. The second derivative of $\rho(z)$ and the first partial derivatives of $c(x,z)$ are continuous.

The source in this problem is concentrated along a line that is parallel to the direction of translational invariance. It generates excitations that are harmonic in time. Therefore, the pressure response of the waveguide can be expressed as the real part of the complex product $\exp(-i\omega t)p(x,z)$, where the positive constant ω is the angular frequency of the source excitation, and the

parameter t is time. A well-known partial differential equation relates the response function p(x,z) to the excitation. For all x and for all z between -H(x) and 0, this equation has the form

$$\frac{\partial^2}{\partial x^2} p(x,z) + \rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial}{\partial z}p(x,z)\right) + K^2(x,z)\,p(x,z)$$

$$= p_s\,\delta(x - x_s)\,\delta(z - z_s)\ , \tag{43}$$

where $K(x,z) = \omega/c(x,z)$. The constant p_s indicates the strength of the source, and the coordinates $x = x_s$ and $z = z_s$ specify its position.

The pressure response is subject to three kinds of boundary conditions in this problem. One of the conditions derives from the vanishing of the normal component of fluid velocity at the rigid bottom. Because the fluid velocity is proportional to the pressure gradient, the normal derivative of the pressure response must vanish at this boundary. Vectors that are normal to the bottom at the position $z = -H(x)$ are parallel to the vector $(\dot{H}(x),1)$, where $\dot{H}(x) = dH(x)/dx$, so this condition can be formulated as

$$\frac{\partial}{\partial z}p(x,-H(x)) + \dot{H}(x)\frac{\partial}{\partial x}p(x,-H(x)) = 0 \tag{44}$$

for all x. The pressure response vanishes at the top boundary, and we note that its tangential derivative vanishes there as well.

$$p(x,0) = 0\ , \tag{45}$$

$$\frac{\partial}{\partial x} p(x,0) = 0 \ , \tag{46}$$

for all x. Finally, the pressure response far from the source must consist essentially of outgoing waves. A mathematical formulation of this condition appears in the last section of this report. It requires that the functions H(x) and c(x,z) converge smoothly to finite limits as |x| tends to infinity.

$$H(x) \to H(\pm\infty) \text{ and } \dot{H}(x) \to 0 \text{ as } x \to \pm\infty, \tag{47}$$

$$c(x,z) \to c(\pm\infty,z) \text{ and } \frac{\partial}{\partial x} c(x,z) \to 0$$

$$\text{uniformly in } z \text{ as } x \to \pm\infty \ . \tag{48}$$

4. ASYMPTOTIC LOCAL ANALYSIS

The eigenvalue problem introduced earlier originates in an asymptotic analysis of self-sustaining oscillations in the waveguide. This analysis produces the functions that we use as a basis for expanding the exact pressure response, and it also suggests the form that the expansion takes. It is of interest, therefore, to give a brief outline of the asymptotic analysis and to discuss some aspects of the expansion functions in this context.

The aim of the analysis is to separate the variations in the pressure response that depend on the range and on the depth. In most cases this can be done only partially, so we adopt the following representation of the pressure response.

$$p(x,z) = a(x)\ \phi(x,z) \ . \tag{49}$$

The factor $a(x)$ is intended to isolate the major range dependent variations in $p(x,z)$. We define the local wavenumber $k(x)$ in terms of this factor.

$$\dot{a}(x) = i\,k(x)\,a(x) \quad , \tag{50}$$

where $\dot{a}(x) = da(x)/dx$. We assume that $k(x)$ and $\phi(x,z)$ are slowly varying functions of range, and in the first stage of this analysis we ignore derivatives of these functions with respect to x when differentiating $p(x,z)$. For example,

$$\frac{\partial}{\partial x} p(x,z) \cong a(x)\,i\,k(x)\,\phi(x,z) \quad , \tag{51}$$

$$\frac{\partial^2}{\partial x^2} p(x,z) \cong -a(x)\,k^2(x)\,\phi(x,z) \quad . \tag{52}$$

These assumptions turn out to be justifiable whenever $|k(x)|\,H(x) \gg 1$ and $|\dot{H}(x)| \ll 1$, although they may be valid under weaker conditions as well.

In terms of the functions $k(x)$ and $\phi(x,z)$, eqs.(43-45) take the following approximate form for self-sustaining oscillations in the waveguide, that is, for $p_s = 0$.

$$\rho(z)\,\frac{\partial}{\partial z}\left\{\frac{1}{\rho(z)}\,\frac{\partial}{\partial z}\,\phi(x,z)\right\} + \left\{K^2(x,z) - k^2(x)\right\}\phi(x,z)$$

$$= 0 \quad , \tag{53}$$

$$\frac{\partial}{\partial z}\,\phi(x,-H(x)) + \dot{H}(x)\,i\,k(x)\,\phi(x,-H(x)) = 0 \quad , \tag{54}$$

$$\phi(x,0) = 0 \quad . \tag{55}$$

Because eqs. (53-55) have the same form as problem (1-3), the set of solutions $k(x) = k_n(x)$ and $\phi(x,z) = \phi_n(x,z)$ is denumerable, and for each x the eigenfunctions $\phi_n(x,z)$ are complete in the two-fold sense described in the principal expansion theorem. Equations (53-55) completely determine the functions $k_n(x)$. These functions are continuously differentiable and slowly varying in that

$$\dot{k}_n(x) \sim -k_n(x)\,\dot{H}(x)/H(x) \quad \text{as } |n| \to \infty , \tag{56}$$

where $\dot{k}_n(x) = dk_n(x)/dx$.

Problem (53-55) does not determine the dependence of the eigenfunctions $\phi_n(x,z)$ on range, however. The next stage of the asymptotic analysis provides this information. We set $k(x) = k_n(x)$ in eq. (50) and then attempt to find a small and slowly varying correction $\Delta_n(x,z)$ that will reduce the difference between the exact factor $\phi(x,z)$ in eq. (49) and the approximation $\phi_n(x,z)$.

$$\phi(x,z) = \phi_n(x,z) + \Delta_n(x,z) , \tag{57}$$

$$|\Delta_n(x,z)| \ll |\phi_n(x,z)| . \tag{58}$$

When we differentiate $p(x,z)$ at this stage, we retain the first derivatives of $k_n(x)$ and $\phi_n(x,z)$ with respect to x, but we ignore the derivatives of $\Delta_n(x,z)$ with respect to x. This results in the following revisions to approximations (51-52).

$$\frac{\partial}{\partial x} p(x,z) \cong a(x) \left\{ i\,k_n(x)\,[\phi_n(x,z) + \Delta_n(x,z)] + \frac{\partial}{\partial x}\phi_n(x,z) \right\} , \tag{59}$$

$$\frac{\partial^2}{\partial x^2} p(x,z) \cong a(x) \left(- k_n^2(x) [\phi_n(x,z) + \Delta_n(x,z)] + 2 i k_n(x) \frac{\partial}{\partial x} \phi_n(x,z) + i \dot{k}_n(x) \phi_n(x,z) \right) . \quad (60)$$

By substituting these approximations in eqs.(43-45), we obtain a nonhomogeneous boundary value problem for each $\Delta_n(x,z)$.

$$\rho(z) \frac{\partial}{\partial z} \left(\frac{1}{\rho(z)} \frac{\partial}{\partial z} \Delta_n(x,z) \right) + \left(K^2(x,z) - k_n^2(x) \right) \Delta_n(x,z)$$

$$= - 2 i k_n(x) \frac{\partial}{\partial x} \phi_n(x,z) - i \dot{k}_n(x) \phi_n(x,z) , \quad (61)$$

$$\frac{\partial}{\partial z} \Delta_n(x,-H(x)) + \dot{H}(x) \, i k_n(x) \Delta_n(x,-H(x))$$

$$= - \dot{H}(x) \frac{\partial}{\partial x} \phi_n(x,-H(x)) , \quad (62)$$

$$\Delta_n(x,0) = 0 . \quad (63)$$

In the present discussion we are interested in the condition that determines whether eqs.(61-63) have solutions, but not in the solutions themselves. This condition is a corollary of the theorem in appendix A; it states that a particular solution of eqs.(61-63) exists if and only if

$$\int_{-H(x)}^{0} \frac{\partial}{\partial x} \left(k_n(x) \phi_n^2(x,z) \right) \frac{\rho(-H(x))}{\rho(z)} dz - \dot{H}(x) \frac{i}{2} \frac{\partial}{\partial x} \phi_n^2(x,-H(x))$$

$$= 0 . \quad (64)$$

Equation (64) is equivalent to a first order ordinary differential equation for the value $\phi_n(x,-H(x))$ on the bottom boundary, and it determines the range dependence of each eigenfunction through this quantity. This condition also causes the eigenfunctions to be slowly varying functions of range in the sense that

$$\frac{\partial}{\partial x}\phi_n(x,z) = \max_{-H(x)\leq z\leq 0} |\phi_n(x,z)|\, O\left(k_n(x)\,\dot{H}(x)\right) . \tag{65}$$

Under the assumptions of the asymptotic analysis, the oscillations described by the functions $k_n(x)$ and $\phi_n(x,z)$ are independent for distinct indices n. This property motivates us to use these functions as the basis of a representation of the exact pressure response in the waveguide problem. We expect the components of such a representation to interact weakly on the whole.

5. EXACT GLOBAL ANALYSIS

In this section we use the collection of solutions $\{k_n(x)\}$ and $\{\phi_n(x,z)\}$ of problem (53-55) to construct an exact representation of the pressure response in the waveguide problem. The form of the representation is suggested by approximations (49) and (51) to $p(x,z)$ and $\partial p(x,z)/\partial x$ in the preceding analysis. The mathematical basis for this representation is the principal eigen-function expansion theorem. The coefficients in the representation turn out to satisfy a linear system of first order ordinary differential equations, and they have jump discontinuities at the position of the line

source. We can integrate these equations to find the coefficients from initial values that describe the asymptotic behavior of $p(x,z)$ far from the source.

In order to develop the expansions and the properties of the coefficients, we first regard the pressure response $p(x,z)$ as known. We let the function $q(x,z)$ denote its partial derivative with respect to range.

$$\frac{\partial}{\partial x} p(x,z) = q(x,z) \quad . \tag{66}$$

This definition makes eq.(43) formally first order in the operator $\partial/\partial x$.

$$\frac{\partial}{\partial x} q(x,z) + \rho(z) \frac{\partial}{\partial z}\left[\frac{1}{\rho(z)} \frac{\partial}{\partial z} p(x,z)\right] + K^2(x,z)\, p(x,z) = p_s\, \delta(x - x_s)\, \delta(z - z_s) \quad . \tag{67}$$

For fixed x, the boundary conditions (44-46) imply that the functions $f(z) = p(x,z)$ and $g(z) = q(x,z)$ satisfy requirements (27-29) of the principal eigenfunction expansion theorem. The derivatives $\partial^3 p(x,z)/\partial z^3$ and $\partial^2 q(x,z)/\partial z^2$ are continuous, except at the source, because the derivatives $\partial c(x,z)/\partial x$, $\partial c(x,z)/\partial z$ and $d^2\rho(z)/dz^2$ are continuous. Therefore, formula (30) uniquely determines the coefficients $a_n(x)$ in the following expansions.

$$p(x,z) = \sum_{n=-\infty}^{\infty}{}' a_n(x)\, \phi_n(x,z) \quad , \tag{68}$$

$$q(x,z) = \sum_{n=-\infty}^{\infty}{}' a_n(x)\, i\, k_n(x)\, \phi_n(x,z) \quad . \tag{69}$$

The expansion of the depth component of the pressure gradient is obtained from expansion (68) by termwise differentiation.

$$\frac{\partial}{\partial z} p(x,z) = \sum_{n=-\infty}^{\infty}{}' a_n(x) \frac{\partial}{\partial z} \phi_n(x,z) . \tag{70}$$

Expansions (68-70) stand for the limits as $N \to \infty$ of the corresponding symmetric partial sums from $n = -N$ to $n = N$, and in this sense they converge uniformly on the closed interval $-H(x) \leq z \leq 0$ if $x \neq x_s$. Expansions (69-70) may be regarded almost as representations of the velocity field in the fluid since the fluid velocity is proportional to the pressure gradient.

We can apply formula (30) to find an expression for each difference $\dot{a}_n(x) - i\,k_n(x)\,a_n(x)$, where $\dot{a}_n(x) = da_n(x)/dx$. Let us define two auxiliary functions $f(x,z)$ and $g(x,z)$ in terms of these differences.

$$f(x,z) = \sum_{n=-\infty}^{\infty}{}' [\dot{a}_n(x) - i\,k_n(x)\,a_n(x)]\,\phi_n(x,z) , \tag{71}$$

$$g(x,z) = \sum_{n=-\infty}^{\infty}{}' [\dot{a}_n(x) - i\,k_n(x)\,a_n(x)]\,i\,k_n(x)\,\phi_n(x,z) . \tag{72}$$

These functions are constructed like expansions (25-26), and their convergence properties are in accord with conditions (37-39) in the uniqueness part of the principal expansion theorem. This is due to the asymptotic relation

$$\dot{a}_n(x) \sim i\, k_n(x)\, a_n(x) \quad \text{as } |n| \to \infty , \tag{73}$$

which generalizes eq.(50) in the preceding asymptotic analysis. Consequently, formula (30) implies that

$$\dot{a}_n(x) - i\, k_n(x)\, a_n(x) = C_n(x) / D_n(x) , \tag{74}$$

where

$$C_n(x) = \int_{-H(x)}^{0} [k_n(x)\, f(x,z) - i\, g(x,z)]\, \phi_n(x,z)\, \frac{\rho(-H(x))}{\rho(z)}\, dz$$

$$- \dot{H}(x)\, i\, f(x,-H(x))\, \phi_n(x,-H(x)) , \tag{75}$$

$$D_n(x) = 2\, k_n(x) \int_{-H(x)}^{0} \phi_n^2(x,z)\, \frac{\rho(-H(x))}{\rho(z)}\, dz$$

$$- \dot{H}(x)\, i\, \phi_n^2(x,-H(x)) . \tag{76}$$

We use eq.(74) to derive the linear system of first order ordinary differential equations for the coefficients. First, we construct new but equivalent expressions for the auxiliary functions $f(x,z)$ and $g(x,z)$ by substituting expansions (68-69) in relations (66-67) and rearranging the resulting sums. The indicated substitutions produce two constraints on the coefficients and their derivatives.

$$\sum_{n=-\infty}^{\infty}{}' \dot{a}_n(x)\, \phi_n(x,z) + \sum_{n=-\infty}^{\infty}{}' a_n(x) \frac{\partial}{\partial x} \phi_n(x,z)$$

$$= \sum_{n=-\infty}^{\infty}{}' a_n(x)\, i\, k_n(x)\, \phi_n(x,z) \;, \tag{77}$$

$$\sum_{n=-\infty}^{\infty}{}' \dot{a}_n(x)\, i\, k_n(x)\, \phi_n(x,z) + \sum_{n=-\infty}^{\infty}{}' a_n(x)\, i\, \dot{k}_n(x)\, \phi_n(x,z)$$

$$+ \sum_{n=-\infty}^{\infty}{}' a_n(x)\, i\, k_n(x) \frac{\partial}{\partial x} \phi_n(x,z) + \sum_{n=-\infty}^{\infty}{}' a_n(x)\, k_n^2(x)\, \phi_n(x,z)$$

$$= p_s\, \delta(x - x_s)\, \delta(z - z_s) \;. \tag{78}$$

Therefore, if $x \neq x_s$ *then the auxiliary functions defined* in eqs.(71-72) have the following equivalent representations.

$$f(x,z) = - \sum_{n=-\infty}^{\infty}{}' a_n(x) \frac{\partial}{\partial x} \phi_n(x,z) \;, \tag{79}$$

$$g(x,z) = - \sum_{n=-\infty}^{\infty}{}' a_n(x)\, i \left(\dot{k}_n(x)\, \phi_n(x,z) + k_n(x) \frac{\partial}{\partial x} \phi_n(x,z) \right) . \tag{80}$$

Substituting these expressions for f(x,z) and g(x,z) in eq.(75) yields an expansion of the factor $C_n(x)$ in terms of the coefficients. This transforms relation (74) into a differential equation that is linear in the coefficients.

If $x \neq x_s$ then

$$\dot{a}_n(x) - i\, k_n(x)\, a_n(x) = - \sum_{m=-\infty}^{\infty}{}' \; a_m(x)\, c_{mn}(x) / D_n(x) \; , \qquad (81)$$

where

$$c_{mn}(x) = [k_m(x) + k_n(x)] \int_{-H(x)}^{0} \left(\frac{\partial}{\partial x} \phi_m(x,z) \right) \phi_n(x,z) \frac{\rho(-H(x))}{\rho(z)} dz$$

$$+ \dot{k}_m(x) \int_{-H(x)}^{0} \phi_m(x,z)\, \phi_n(x,z) \frac{\rho(-H(x))}{\rho(z)} dz$$

$$- \dot{H}(x)\, i \left(\frac{\partial}{\partial x} \phi_m(x,-H(x)) \right) \phi_n(x,-H(x)) \qquad (82)$$

for $m,n = \pm 1, \pm 2$, and so on.

It is interesting to note some structural features of these equations. First, eq.(81) has the same form as standard coupled mode equations associated with waveguides that have parallel plane boundaries. On the other hand, eqs.(76) and (82) differ from standard expressions because they include terms that are proportional to $\dot{H}(x)$. In the derivation of these equations we have not used condition (64), which determines the range dependence of the eigenfunctions. If this condition is adopted, however, it causes the diagonal matrix elements $c_{nn}(x)$ to vanish.

If $x = x_s$ then the derivatives $\dot{a}_n(x)$ fail to exist because each coefficient $a_n(x)$ has a jump discontinuity at this range. We calculate the jumps as follows.

$$a_n(x_s + 0) - a_n(x_s - 0)$$
$$= \lim_{\varepsilon \to 0^+} [a_n(x_s + \varepsilon) - a_n(x_s - \varepsilon)] \quad . \tag{83}$$

A pair of constraints on these jump discontinuities can be obtained by integrating eqs.(77-78) from $x = x_s - \varepsilon$ to $x = x_s + \varepsilon$ and then letting $\varepsilon \to 0^+$ in the result.

$$\sum_{n=-\infty}^{\infty}{}' \; [a_n(x_s + 0) - a_n(x_s - 0)] \, \phi_n(x_s, z) = 0 \quad , \tag{84}$$

$$\sum_{n=-\infty}^{\infty}{}' \; [a_n(x_s + 0) - a_n(x_s - 0)] \, i \, k_n(x_s) \, \phi_n(x_s, z)$$
$$= p_s \, \delta(z - z_s) \quad . \tag{85}$$

Because these constraints have the same form as eqs.(25-26), we can evaluate the jump discontinuities by another application of formula (30) with $f(z) = 0$ and $g(z) = p_s \, \delta(z - z_s)$.

$$a_n(x_s + 0) - a_n(x_s - 0) = - p_s \frac{i \, \phi_n(x_s, z_s)}{D_n(x_s)} \frac{\rho(-H(x_s))}{\rho(z_s)} \quad . \tag{86}$$

Although the pressure response has been regarded as known up to this point, it is in fact initially unknown in the waveguide problem. Reversing the direction of the preceding discussion leads to a method for finding p(x,z) in this problem. First, we find the coefficients $a_n(x)$ by integrating eqs. (81) subject to conditions (86), and then we use the coefficients to construct expansion (68) for p(x,z). In order to start the integration, however, we need values of the coefficients at some reference points. Our choices of the initial conditions are motivated by the requirement that the pressure response should consist essentially of outgoing waves far from the source.

Under assumptions (47-48) of the waveguide problem, the functions $a_n(x)$ turn out to be basically exponential far from the source. Each $k_n(x)$ tends to purely real or purely imaginary limits $k_n(\pm\infty)$ as $x \to \pm\infty$, and there are complex amplitudes $A_n(\pm\infty)$ such that

$$a_n(x) \exp\{- i\, k_n(\pm\infty)\, x\} \to A_n(\pm\infty) \quad \text{as } x \to \pm\infty \; . \tag{87}$$

Due to the underlying $\exp(-i\omega t)$ time dependence of the pressure response, the following initial conditions at the indicated ranges exclude propagating waves that approach the source from infinity. These conditions also exclude stationary oscillations that grow unboundedly in amplitude as $|x|$ increases without limit.

$$a_n(-\infty) = 0 \text{ if } \mathrm{Re}\{k_n(-\infty)\} > 0 \text{ or } \mathrm{Im}\{k_n(-\infty)\} > 0 \; , \tag{88}$$

$$a_n(+\infty) = 0 \text{ if } \mathrm{Re}\{k_n(+\infty)\} < 0 \text{ or } \mathrm{Im}\{k_n(+\infty)\} < 0 \; . \tag{89}$$

APPENDIX A: A NONHOMOGENEOUS BOUNDARY VALUE PROBLEM

In this appendix we state a necessary and sufficient condition for the existence of a particular solution $\Phi(z)$ to the following nonhomogeneous boundary value problem.

$$\rho(z) \frac{d}{dz}\left(\frac{1}{\rho(z)} \frac{d}{dz} \Phi(z)\right) + \left(K^2(z) - k_n^2\right) \Phi(z) = F(z)$$

$$\text{for } -H < z < 0 \ , \tag{A1}$$

$$\frac{d}{dz} \Phi(-H) + \dot{H}\, i\, k_n \Phi(-H) = -\dot{H}\gamma \ , \tag{A2}$$

$$\Phi(0) = 0 \ . \tag{A3}$$

In this problem, k_n is an eigenvalue of problem (1-3). We assume that the function $F(z)$ has a continuous second derivative on the closed interval $-H \leq z \leq 0$ and that $F(0)$ vanishes. The parameter γ may take any complex value.

Theorem.

Problem (A1-A3) has a solution if and only if

$$\int_{-H}^{0} F(z)\phi_n(z) \frac{\rho(-H)}{\rho(z)} dz - \dot{H}\gamma\phi_n(-H) = 0 \ , \tag{A4}$$

where the eigenfunction $\phi_n(z)$ is associated with k_n. In addition, if relation (20) holds for all indices, then one particular solution can be represented as follows.

$$\Phi(z) = \sum_{\substack{m=-\infty \\ (m \neq n)}}^{\infty}{}' (k_m - k_n)^{-1} a_m \phi_m(z) \ , \tag{A5}$$

$$a_m = \frac{\displaystyle\int_{-H}^{0} F(z)\,\phi_m(z)\,\frac{\rho(-H)}{\rho(z)}\,dz \; - \; \dot{H}\,\gamma\,\phi_m(-H)}{\displaystyle 2\,k_m \int_{-H}^{0} \phi_m^2(z)\,\frac{\rho(-H)}{\rho(z)}\,dz \; - \; \dot{H}\,i\,\phi_m^2(-H)} \; . \tag{A6}$$

It is easy to prove the necessity of condition (A4). As in the derivation of relation (19), we first express eq.(1) for $\phi_n(z)$ and eq.(A1) for $\Phi(z)$ in the respective operator forms $L(\phi_n,k_n) = 0$ and $L(\Phi,k_n) - F = 0$. Then we obtain the following identity by partial integration of the difference $(\Phi/\rho)L(\phi_n,k_n) - (\phi_n/\rho)L(\Phi,k_n)$:

$$\int_{-H}^{0} F(z)\,\phi_n(z)\,\frac{1}{\rho(z)}\,dz$$

$$= \left\{ \phi_n(z)\,\frac{1}{\rho(z)}\,\frac{d}{dz}\,\Phi(z) - \Phi(z)\,\frac{1}{\rho(z)}\,\frac{d}{dz}\,\phi_n(z) \right\}_{z=-H}^{z=0} . \tag{A7}$$

This identity leads directly to condition (A4) after we apply boundary conditions (2-3) for $\phi_n(z)$ and boundary conditions (A2-A3) for $\Phi(z)$.

One can prove the sufficiency of condition (A4) by verifying that eqs.(A5-A6) represent a particular solution of problem (A1-A3) when it holds. This proof requires use of the principal eigenfunction expansion theorem, but it will not be given here. Equations (A5-A6) are provided only to indicate the form of the correction terms $\Delta_n(x,z)$ in the asymptotic analysis of the waveguide problem.